Volume elements of spacetime and a quartet of scalar fields

Frank Gronwald[1], Uwe Muench[1], Alfredo Macías[2], Friedrich W. Hehl[1]
[1] *Institute for Theoretical Physics, University of Cologne, D-50923 Köln, Germany*
[2] *Departamento de Física, Universidad Autónoma Metropolitana-Iztapalapa, P.O. Box 55-534, 09340 México D.F., México*

(received ...)

Starting with a 'bare' 4-dimensional differential manifold as a model of spacetime, we discuss the options one has for defining a volume element which can be used for physical theories. We show that one has to prescribe a scalar density $\boldsymbol{\sigma}$. Whereas conventionally $\sqrt{|\det g_{ij}|}$ is used for that purpose, with g_{ij} as the components of the metric, we point out other possibilities, namely $\boldsymbol{\sigma}$ as a 'dilaton' field or as a derived quantity from either a linear connection or a quartet of scalar fields, as suggested by Guendelman and Kaganovich. *file volume5a.tex, 1997-12-15*

PACS numbers: 04.50.+h, 02.30.Cj, 04.20.Fy, 12.10.-g

I. INTRODUCTION

A fundamental premise is that gravity is intimately intertwined with the geometry of spacetime. At the classical level, general relativity captures this idea in an elegant way. The identification of the gravitational field with the curvature of spacetime has led to the most dramatic predictions of general relativity, such as the existence of black holes and the occurrence of the big bang. This leads to the viewpoint that a primary goal of any quantum theory should be the quantum structure of spacetime.

There are two programs which have made great strides forward into the quantum gravity in the last twelve years, namely, one perturbative, i.e. *the quantum superstring*, and one nonperturbative, i.e. *Dirac's canonical quantization* program, applying Ashtekar realization of Einsteins's general relativity and the loop representation for quantum mechanics.

At the first sight, it would appear that the two approaches are incompatible. Ashtekar's program applies pure Einstein gravity, without involving anything else, whereas the superstring is a *theory of everything* and, because of unitarity, would involve all interactions in its perturbative expansion. These differences will appear at Planck energies only, and only around these energies one might be able to decide between string theory and canonical gravity.

On the other hand, the study of gravitational interactions coupled to Maxwell and scalar dilaton fields has been the subject of recent investigations. Dilaton fields appear naturally in the low energy limit of string theory. Therefore, the study of the scalar fields is of importance for the understanding of more general theories. Moreover, it has been demostrated that the metric-affine gravity theories contain the axi-dilatonic sector of low energy string theory. The gravitational interactions involving the axion and dilaton may be derived from a geometrical action principle involving the curvature scalar with a non-Riemannian connection, i.e. the axi-dilatonic sector of the low energy string theory can be expressed in terms of geometry with torsion and nonmetricity [1].

The local geometry of spacetime is usually characterized by two independent concepts: The concept of a linear connection (parallel transport) and the concept of a metric (length and angle measurements). Both, the linear connection and the metric are physical fields which have to be determined from field equations of the gravitational theory under consideration.

In physics we do not only want to postulate the fields that are needed in order to formulate a theory. We also find it desirable to explain the existence of these fields by means of some fundamental principle. Therefore we are led to wonder what fundamental principles would suggest the existence of the linear connection and the metric, respectively.

The existence of the *linear connection* is quite satisfactorily explained by a symmetry principle which is believed to be fundamental: This is the principle of local gauge invariance. In the case of gravity we focus on *external* symmetries, i.e., symmetry transformations of spacetime. Local gauge invariance then reflects the invariance of a physical system under such transformations. As in the Yang-Mills theories, this requires the introduction of a gauge connection which, in turn, allows to define parallel transport. The actual gauge connection depends on the specific symmetry under consideration and is, in a gauge approach to gravity, of the form of a linear connection together with the coframe; for details, see [2,3].

In contrast to this, it is not clear how to derive the *metric* from some fundamental principle. Usually the existence of the metric is simply assumed, sometimes in disguise of a local symmetry group which contains an orthogonal subgroup. Therefore it is quite natural to ask whether the metric itself is a fundamental quantity, a derived quantity, or a quantity which can be substituted by some more fundamental field. To investigate this question is one of the motivations for this article. Another motivation is our desire to understand the physical con-

sequences of the resulting theories and whether or not they are more suitable for quantization. As we will see, both motivations will lead to interrelated questions and structures.

Our starting point is the observation that the metric is commonly taken to define a volume element in order to be able to perform integrations on spacetime. Integrated objects are clearly of fundamental importance in physics. However, the definition of a volume element on spacetime is also possible without reference to any metric. This general subject will be explained in Sec. II. Basically, a volume element can be defined on any differentiable manifold as the determinant of a parallelepiped defined in terms of n vectors, if n is the dimension of the manifold. Then no *absolute* volume measure exists. However, *proportions* of different volumes can be determined. More explicitly, one finds that the volume element is the Levi-Civita n-form density transvected with the components of the n linearly independent vectors spanning the parallelepiped. Such a volume is an (odd) density of weight -1. In order to define an integral, we need then additionally a scalar density of weight $+1$.

Usual physical fields are no densities. Therefore the common practice is to take the metric and to build a density according to $\sqrt{|\det g_{ij}|}$. But there exist alternatives, as we will point out in Sec. III. They open the gate to alternative theories of gravitation.

Subsequently, in Sec. IV, we will follow up one possibility, namely the quartet of scalar fields, as proposed in [4–6]. In an appendix, we will provide some mathematical background for the differentiation of some quantities closely related to the volume element.

II. INTEGRATION ON SPACETIME

We model spacetime as a 4-dimensional differentiable manifold, which is assumed to be paracompact, Hausdorff, and connected. We will restrict ourselves to four dimensions. Generalization to arbitrary dimensions is straightforward. In order to be able to formulate physical laws on such a spacetime, we have to come up with suitably defined integrals. If we want, for example, to specify a *scalar* action functional W of a physical system,

$$W = \int L = \int \widetilde{\epsilon}\, \widehat{L}\ , \tag{2.1}$$

then, taking the integral in its conventional (Lebesgue) meaning, the Lagrangian L has to be an odd 4-form in order to make the integral (2.1) really a scalar. Incidentally, a p-form $\omega = \frac{1}{p!}\,\omega_{i_1\dots i_p}\,dx^{i_1}\wedge\dots\wedge dx^{i_p}$ is called *even* if it is invariant under a diffeomorphism $x^i \to x'^i(x^j)$ with $\det\big(\partial x^j/\partial x'^i\big) < 0$. It is called *odd* if it changes its sign under such a diffeomorphism; for even and odd forms, see Burke [7], Bott and Tu [8], and also [2].

Now, any odd 4-form can be split into a product of a 0-form (or scalar) $\widehat{L}$ and another odd 4-form $\widetilde{\epsilon}$. We assume that we did the splitting in such a way that the properties of the physical system are subsumed in the scalar $\widehat{L}$ and the properties of spacetime in the *odd volume 4-form* $\widetilde{\epsilon}$. In the case of *gravity* such a distinction may be ambiguous, because the properties of spacetime themselves are parts of the physical system.

Let us consider a trivial physical system with $\widehat{L} = 1$. Then the integral measures the volume of the corresponding piece of spacetime,

$$\mathrm{Vol} = \int \widetilde{\epsilon}\,. \tag{2.2}$$

For that reason $\widetilde{\epsilon}$ is called a volume form or, more colloquially, a *volume element* of spacetime. This quantity can be split again into two pieces.

As the first piece we have the Levi-Civita $\boldsymbol{\epsilon}$ in mind which is a very special geometric object. The Levi-Civita $\boldsymbol{\epsilon}$ can be defined on any differential manifold. It is the 'purely geometrical' volume element (see Grassmann's theory of extension [*Ausdehnungslehre*] or the discussion of Laurent [9]). In order to define $\boldsymbol{\epsilon}$, we recall that, besides the components of the Kronecker symbol δ_i^j, the components $\boldsymbol{\epsilon}_{ijkl}$ of the Levi-Civita $\boldsymbol{\epsilon}$ are, by assumption, *numerically invariant* under diffeomorphisms. There exist no other quantities of this kind, apart from $\boldsymbol{\epsilon}^{ijkl}$ of Sec. III D. And, in this sense, these components are very special. Now we define $\boldsymbol{\epsilon}$ in terms of its components:

$$\boldsymbol{\epsilon} := \frac{1}{4!}\,\boldsymbol{\epsilon}_{ijkl}\,dx^i\wedge dx^j\wedge dx^k\wedge dx^l\ , \qquad \text{with}\quad \boldsymbol{\epsilon}_{0123} = 1 = \text{invariant}\ . \tag{2.3}$$

Consequently $\boldsymbol{\epsilon}$ transforms as an *odd 4-form density of weight* -1 (see, e.g., [2, Appendix A], for details),

$$\boldsymbol{\epsilon}' = \frac{1}{J}\,\boldsymbol{\epsilon} = \frac{\mathrm{sgn}J}{|J|}\,\boldsymbol{\epsilon}\ , \tag{2.4}$$

where $J = \det\big(\partial x^j/\partial x'^i\big)$ is the determinant of the Jacobian matrix of the diffeomorphism $x^i \to x'^i(x^j)$. We are denoting densities by boldface letters.

We note in passing that for the manipulation of the $\boldsymbol{\epsilon}$-basis the following algebraic rules will turn out to be useful. If we take the interior product $\rfloor$ of an arbitrary frame e_α with the Levi-Civita $\boldsymbol{\epsilon}$ 4-form density, then we find a 3-form $\boldsymbol{\epsilon}_\alpha$; if we contract again, we find a 2-form $\boldsymbol{\epsilon}_{\alpha\beta}$, etc.:

$$\boldsymbol{\epsilon}_\alpha := e_\alpha\rfloor\boldsymbol{\epsilon} = \frac{1}{3!}\,\boldsymbol{\epsilon}_{\alpha\beta\gamma\delta}\,\vartheta^\beta\wedge\vartheta^\gamma\wedge\vartheta^\delta\ , \tag{2.5a}$$

$$\boldsymbol{\epsilon}_{\alpha\beta} := e_\beta\rfloor\boldsymbol{\epsilon}_\alpha = \frac{1}{2!}\,\boldsymbol{\epsilon}_{\alpha\beta\gamma\delta}\,\vartheta^\gamma\wedge\vartheta^\delta\ , \tag{2.5b}$$

$$\boldsymbol{\epsilon}_{\alpha\beta\gamma} := e_\gamma\rfloor\boldsymbol{\epsilon}_{\alpha\beta} = \frac{1}{1!}\,\boldsymbol{\epsilon}_{\alpha\beta\gamma\delta}\,\vartheta^\delta\ , \tag{2.5c}$$

$$\boldsymbol{\epsilon}_{\alpha\beta\gamma\delta} \ = e_\delta\rfloor\boldsymbol{\epsilon}_{\alpha\beta\gamma} = e_\delta\rfloor e_\gamma\rfloor e_\beta\rfloor e_\alpha\rfloor\boldsymbol{\epsilon}\ . \tag{2.5d}$$

Here, the coframe ϑ^β is dual to the frame e_α, that is, $e_\alpha \rfloor \vartheta^\beta = \delta^\beta_\alpha$. The $(\boldsymbol{\epsilon}\,, \boldsymbol{\epsilon}_\alpha\,, \boldsymbol{\epsilon}_{\alpha\beta}\,, \boldsymbol{\epsilon}_{\alpha\beta\gamma}\,, \boldsymbol{\epsilon}_{\alpha\beta\gamma\delta})$ represent a basis for the odd form densities of weight -1. It is called $\boldsymbol{\epsilon}$-basis and can be used to define a metric independent duality operation. Instead of lowering the rank of the $\boldsymbol{\epsilon}$'s, we can also increase their rank by exterior multiplication with the coframe ϑ^μ:

$$\vartheta^\mu \wedge \boldsymbol{\epsilon}_\alpha = +\delta^\mu_\alpha\, \boldsymbol{\epsilon}\,, \tag{2.6a}$$
$$\vartheta^\mu \wedge \boldsymbol{\epsilon}_{\alpha\beta} = -\delta^\mu_\alpha\, \boldsymbol{\epsilon}_\beta + \delta^\mu_\beta\, \boldsymbol{\epsilon}_\alpha\,, \tag{2.6b}$$
$$\vartheta^\mu \wedge \boldsymbol{\epsilon}_{\alpha\beta\gamma} = +\delta^\mu_\alpha\, \boldsymbol{\epsilon}_{\beta\gamma} - \delta^\mu_\beta\, \boldsymbol{\epsilon}_{\alpha\gamma} + \delta^\mu_\gamma\, \boldsymbol{\epsilon}_{\alpha\beta}\,, \tag{2.6c}$$
$$\vartheta^\mu \wedge \boldsymbol{\epsilon}_{\alpha\beta\gamma\delta} = -\delta^\mu_\alpha\, \boldsymbol{\epsilon}_{\beta\gamma\delta} + \delta^\mu_\beta\, \boldsymbol{\epsilon}_{\alpha\gamma\delta} - \delta^\mu_\gamma\, \boldsymbol{\epsilon}_{\alpha\beta\delta} + \delta^\mu_\delta\, \boldsymbol{\epsilon}_{\alpha\beta\gamma}\,. \tag{2.6d}$$

For the $\widetilde{\epsilon}$ (which is an odd 4-form density of weight 0), formulae analogous to (2.5),(2.6) are valid. We have just to add twiddles to the ϵ's.

Since $\boldsymbol{\epsilon}$ is an odd density of weight -1, we can split the volume element $\widetilde{\epsilon}$, if we postulate the existence of an *even scalar density* $\boldsymbol{\sigma}$ of weight $+1$, that is,

$$\boldsymbol{\sigma}' = |J|\, \boldsymbol{\sigma}\,. \tag{2.7}$$

For our purpose here[1], we postulated an *even* scalar density, since the $\widetilde{\epsilon}$ in (2.1) and the Levi-Civita $\boldsymbol{\epsilon}$ in (2.3) are both odd. Then eventually, equation (2.1) can be rewritten as

$$\begin{aligned} W &= \int \underbrace{L}_{\text{odd 4-f.}} = \int \underbrace{\widetilde{\epsilon}}_{\text{odd 4-f.}}\, \underbrace{\widehat{L}}_{\text{scalar}} \\ &= \int \underbrace{\boldsymbol{\epsilon}}_{\substack{\text{odd 4-f. density,}\\ \text{weight } -1}}\, \underbrace{\boldsymbol{\sigma}}_{\substack{\text{even scalar density,}\\ \text{weight } +1}}\, \underbrace{\widehat{L}}_{\text{scalar}}\,. \end{aligned} \tag{2.8}$$

The scalar density $\boldsymbol{\sigma}$, in contrast to the Levi-Civita $\boldsymbol{\epsilon}$, must be specified in some way, before one can actually do physics on the spacetime manifold. It is here where gravity comes in.

III. CHOICES FOR THE SCALAR DENSITY $\boldsymbol{\sigma}$

A. Metric

It is conventional wisdom to choose the square root of the modulus of the metric determinant as the scalar density for building up the volume element:

$${}_{\mathbf{0}}\boldsymbol{\sigma} := \sqrt{|\det g_{ij}|}\,. \tag{3.1}$$

As soon as a metric $g = g_{ij}\, dx^i \otimes dx^j$ is given—the gravitational potential of general relativity—we can define ${}_{\mathbf{0}}\boldsymbol{\sigma}$. In conventional integration theory, this is called the volume measure. We prefer to call it the metric volume measure and, accordingly, $\eta := {}_{\mathbf{0}}\boldsymbol{\sigma}\, \boldsymbol{\epsilon}$ the *metric volume element*. Remember that Einstein, in his 1916 review paper of general relativity, see [10, p. 304], only admitted coordinates such that ${}_{\mathbf{0}}\boldsymbol{\sigma} \stackrel{*}{=} 1$. This amounted to use the 'purely geometrical' volume element $\boldsymbol{\epsilon}$ by constraining the free choice of the coordinates. It should be noted, however, that Einstein, in formulating general relativity, did *not* restrict the diffeomorphism invariance in any way, as he stressed himself (loc.cit.), to what is sometimes called a volume-preserving diffeomorphism (see [11–13]). For him, it was only a convenience in evaluating the diffeomorphism covariant equations of general relativity.

B. Dilaton field

Alternatively, we can promote the scalar density to a new fundamental field of nature, compare also the model developed in [2, Sec.6]. The value of such a density ${}_{\mathbf{1}}\boldsymbol{\sigma}$ can be viewed as a scale factor of the volume element, see also [11,13]. Thus, from a physical point of view, it is interesting to investigate the role of ${}_{\mathbf{1}}\boldsymbol{\sigma}$ as a scaling parameter which realizes a scale transformation on a physical system.

Scale transformations as symmetry transformations play an important part in physics. In particular in the context of cosmological models and the study of the unification of the four interactions it is common to start from a scale invariant theory (for the sake of renormalizability, e.g.) and later derive the scales nowadays observed by some mechanism like spontaneous symmetry breaking.

Clearly, scale invariance has to be carefully defined—and there are several possibilities. Well-known scale transformations are the Weyl transformations $g_{ij} \longrightarrow e^{2\Phi} g_{ij}$ which can be thought of as a rescaling of the metric. In conventional theories, where the volume element is taken as $\sqrt{|\det g_{ij}|}$, a Weyl transformation scales the volume element in a definite way. This is important if Weyl-invariance of an action integral is considered as it is the case in string theories. The possibility to take a metric independent scalar density ${}_{\mathbf{1}}\boldsymbol{\sigma}$ in place of $\sqrt{|\det g_{ij}|}$ in order to build a proper volume element gives new opportunities to define a concept of scale invariance. In this context, the field ${}_{\mathbf{1}}\boldsymbol{\sigma}$ is known as a dilaton field, which becomes non-trivial in the quantum theory after the conformal (scale) invariance is broken.

For example, a replacement of $\sqrt{|\det g_{ij}|}$ by an independent ${}_{\mathbf{1}}\boldsymbol{\sigma}$ will, a priori, 'decouple' the scaling properties of the volume element from the scaling properties

[1] In reference [2, Eq.(A.1.33)] we took an *odd* scalar density instead, which we also denoted by the same letter σ.

of the (gravitational and matter) Lagrangian. Moreover, the gravitational field equations which are obtained from varying the metric, will, in general, be changed. In particular, the metrical energy momentum current will take a modified form, due to the absence of the factor $\sqrt{|\det g_{ij}|}$. Here, the possibility of introducing the independent field ${}_{\mathbf{1}}\boldsymbol{\sigma}$ can lead to a modification of this current and (possibly) related anomalies.[2] In this sense, employing ${}_{\mathbf{1}}\boldsymbol{\sigma}$ as independent scale parameter yields a lot of opportunities for physical creativity. We will come back to this question elsewhere.

C. Linear connection

In a pure connection ansatz, we prescribe a linear connection $\Gamma_\alpha{}^\beta = \Gamma_{i\alpha}{}^\beta dx^i$ (but *no* metric!). Define, as usual, the curvature-2-form by

$$R_\alpha{}^\beta = d\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta \tag{3.2}$$

and the Ricci-1-form by

$$\mathrm{Ric}_\alpha := e_\beta \rfloor R_\alpha{}^\beta = \mathrm{Ric}_{i\alpha}\, dx^i \; . \tag{3.3}$$

Then

$$ {}_{\mathbf{2}}\boldsymbol{\sigma} := \sqrt{|\det \mathrm{Ric}_{ij}|}\; , \tag{3.4}$$

with $\mathrm{Ric}_{ij} = \mathrm{Ric}_{i\alpha} e_j{}^\alpha$, is a viable scalar density, as first suggested by Eddington [14, Sec. 92], compare also Schrödinger [15] and the more recent work of Kijowski and collaborators, see [16]. Likewise, the corresponding quantity based on the *symmetric* part of the Ricci tensor,

$$ {}_{\mathbf{3}}\boldsymbol{\sigma} := \sqrt{\left|\det \mathrm{Ric}_{(ij)}\right|}\; , \tag{3.5}$$

also qualifies as a volume measure. Note that ${}_{\mathbf{2}}\boldsymbol{\sigma}$ and ${}_{\mathbf{3}}\boldsymbol{\sigma}$ may have singular points in such a theory as soon as the Ricci tensor or its symmetric part vanish. There seems to exist no criterion around which would prefer, say, ${}_{\mathbf{2}}\boldsymbol{\sigma}$, as compared to ${}_{\mathbf{3}}\boldsymbol{\sigma}$. Lately, such theories have been abandoned.

D. A quartet of scalar fields

In close analogy to the components $\boldsymbol{\epsilon}_{ijkl}$ of the Levi-Civita $\boldsymbol{\epsilon}$, we can define the totally antisymmetric tensor density $\boldsymbol{\epsilon}^{ijkl}$ of weight $+1$. We put its numerically invariant component $\boldsymbol{\epsilon}^{0123} = -1$.

We then can define [4]

$$\begin{aligned} {}_{\mathbf{4}}\boldsymbol{\sigma} &:= -\boldsymbol{\epsilon}^{ijkl} \left(\partial_i\varphi^{(0)}\right)\left(\partial_j\varphi^{(1)}\right)\left(\partial_k\varphi^{(2)}\right)\left(\partial_l\varphi^{(3)}\right) \\ &= -\frac{1}{4!}\,\boldsymbol{\epsilon}^{ijkl}\,\epsilon_{ABCD}\,\left(\partial_i\varphi^A\right)\left(\partial_j\varphi^B\right)\left(\partial_k\varphi^C\right)\left(\partial_l\varphi^D\right)\; , \end{aligned} \tag{3.6}$$

where $A,\dots,D$ are indices of interior space. This definition yields, for the volume 4-form

$$\overline{\eta} := {}_{\mathbf{4}}\boldsymbol{\sigma}\,\boldsymbol{\epsilon}\; , \tag{3.7}$$

the following relations:

$$\begin{aligned} \overline{\eta} &= d\varphi^{(0)} \wedge d\varphi^{(1)} \wedge d\varphi^{(2)} \wedge d\varphi^{(3)} \\ &= \frac{1}{4!}\,\epsilon_{ABCD}\, d\varphi^A \wedge d\varphi^B \wedge d\varphi^C \wedge d\varphi^D\; . \end{aligned} \tag{3.8}$$

If we introduce the abbreviation

$$\partial_A := \frac{\partial}{\partial\varphi^A}\; , \tag{3.9}$$

then the duality of $d\varphi^A$ and ∂_B can be expressed as follows:

$$d\varphi^A\,[\partial_B] = \delta^A_B\; . \tag{3.10}$$

In analogy to the set of equation (2.5), we define the 3-form and the 2-form

$$\overline{\eta}_A := \partial_A \rfloor \overline{\eta}\; , \quad \overline{\eta}_{AB} := \partial_B \rfloor \overline{\eta}_A\; , \quad \text{etc.} \tag{3.11}$$

Explicitly they read

$$\overline{\eta}_A = \frac{1}{3!}\epsilon_{ABCD} d\varphi^B \wedge d\varphi^C \wedge d\varphi^D\; , \tag{3.12a}$$

$$\overline{\eta}_{AB} = \frac{1}{2!}\epsilon_{ABCD} d\varphi^C \wedge d\varphi^D\; , \quad \text{etc.} \tag{3.12b}$$

In analogy to (2.6) we have

$$d\varphi^N \wedge \overline{\eta}_A = +\delta^N_A\,\overline{\eta}\; , \quad d\varphi^N \wedge \overline{\eta}_{AB} = -\delta^N_A\,\overline{\eta}_B + \delta^N_B\,\overline{\eta}_A\; , \tag{3.13}$$

and so on. We contract (3.13) and find

$$\overline{\eta} = \frac{1}{4}\, d\varphi^N \wedge \overline{\eta}_N\; , \quad \overline{\eta}_A = \frac{1}{3}\, d\varphi^N \wedge \overline{\eta}_{AN}\; , \quad \text{etc.} \tag{3.14}$$

We differentiate (3.14):

$$d\,\overline{\eta} = -\frac{1}{4}\, d\varphi^N \wedge d\,\overline{\eta}_N\; , \quad d\,\overline{\eta}_A = -\frac{1}{3}\, d\varphi^N \wedge d\,\overline{\eta}_{AN}\; , \quad \text{etc.} \tag{3.15}$$

[2] In this respect the computations by Buchmüller and Dragon [12] concerning the quantization of gravitation in the volume-preserving gauge ${}_{\mathbf{0}}\boldsymbol{\sigma} \overset{*}{=} 1$ should be useful as a first ansatz.

Now, $\overline{\eta}$, as a 4-form, is closed:

$$d\,\overline{\eta} = 0\,. \tag{3.16}$$

Provided $d\varphi^A \neq 0$, we find successively,

$$d\,\overline{\eta}_A = 0\,, \quad d\,\overline{\eta}_{AB} = 0\,, \quad \text{etc.} \tag{3.17}$$

Using this information, we can partially integrate (3.14) and can prove that all these forms are not only closed, but also exact:

$$\overline{\eta} = d\left[\frac{1}{4}\,\varphi^N \wedge \overline{\eta}_N\right]\,, \quad \overline{\eta}_A = d\left[\frac{1}{3}\,\varphi^N \wedge \overline{\eta}_{AN}\right]\,, \quad \text{etc.} \tag{3.18}$$

Using (3.8) and (3.12), we find

$$\frac{\partial\,\overline{\eta}}{\partial\,d\varphi^A} = \overline{\eta}_A\,, \quad \frac{\partial\,\overline{\eta}_A}{\partial\,d\varphi^B} = \overline{\eta}_{AB}\,, \quad \text{etc.} \tag{3.19}$$

or, because of (3.18):

$$d\,\frac{\partial\,\overline{\eta}}{\partial\,d\varphi^A} = 0\,, \quad d\,\frac{\partial\,\overline{\eta}_A}{\partial\,d\varphi^B} = 0\,, \quad \text{etc.} \tag{3.20}$$

Since the corresponding "forces" vanish too, as can be seen from (3.8) and (3.12),

$$\frac{\partial\,\overline{\eta}}{\partial\,\varphi^A} = 0\,, \quad \frac{\partial\,\overline{\eta}_A}{\partial\,\varphi^B} = 0\,, \quad \text{etc.,} \tag{3.21}$$

we find an analogous result for the variational derivatives:

$$\frac{\delta\,\overline{\eta}}{\delta\,\varphi^A} = 0\,, \quad \frac{\delta\,\overline{\eta}_A}{\delta\,\varphi^B} = 0\,, \quad \text{etc.} \tag{3.22}$$

Similarly, we have

$$\frac{\delta\overline{\eta}}{\delta\vartheta^\alpha} = 0\,, \quad \frac{\delta\overline{\eta}_A}{\delta\vartheta^\alpha} = 0\,, \quad \text{etc.} \tag{3.23}$$

and

$$\frac{\delta\overline{\eta}}{\delta g_{\alpha\beta}} = 0\,, \quad \frac{\delta\overline{\eta}_A}{\delta g_{\alpha\beta}} = 0\,, \quad \text{etc.} \tag{3.24}$$

That the volume element is an exact form is the distinguishing feature of this ansatz. It is for that reason why the existence of a quartet of fundamental scalar fields is required instead of only one scalar field. Under these circumstances, the volume (2.2) can be expressed, via Stokes' theorem, as a 3-dimensional surface integral which doesn't contribute to the variation of the action functional.

IV. THE QUARTET THEORY

Using Eq. (2.8) and the volume element (3.8) and denoting the gravitational Lagrangian by $V = V(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R_\alpha{}^\beta)$ and the matter Lagrangian by $L_{\rm m} = L_{\rm m}(g_{\alpha\beta}, \vartheta^\alpha, \Gamma_\alpha{}^\beta, \Psi, d\Psi)$, the action W reads

$$W = \int (V + L_{\rm m}) = \int \overline{\eta}\; \underbrace{(\widehat{V} + \widehat{L}_{\rm m})}_{\text{scalar}} = \int d\,\chi\,(\widehat{V} + \widehat{L}_{\rm m})\,, \tag{4.1}$$

see Guendelman and Kaganovich [4–6]. Note that the 3-form χ, according to (3.18), explicitly reads $\chi := \varphi^N \wedge \overline{\eta}_N/4$. If we add a constant λ to the scalar Lagrangian, we find

$$\int d\,\chi\,(\widehat{V} + \widehat{L}_{\rm m} + \lambda) = W + \lambda \int d\,\chi\,. \tag{4.2}$$

Since the 3-dimensional hypersurface integral $\int_{\partial\,\rm Vol}\chi$ doesn't contribute to the variation, the scalar Lagrangian is invariant under the addition of a constant.

Variation with respect to φ^A yields the corresponding field equations

$$\frac{\partial\,(V + L_{\rm m})}{\partial\varphi^A} - d\,\frac{\partial\,(V + L_{\rm m})}{\partial\,d\varphi^A} = 0\,. \tag{4.3}$$

Suppose, see [4–6], that $\widehat{V}$ and $\widehat{L}_{\rm m}$ do *not* depend on the quartet field at all,

$$\frac{\partial\,\widehat{V}}{\partial\,\varphi^A} = 0\,, \quad \frac{\partial\,\widehat{V}}{\partial\,d\varphi^A} = 0\,, \quad \frac{\partial\,\widehat{L}_{\rm m}}{\partial\,\varphi^A} = 0\,, \quad \frac{\partial\,\widehat{L}_{\rm m}}{\partial\,d\varphi^A} = 0\,, \tag{4.4}$$

then the field equations for the quartet field read

$$(\widehat{V} + \widehat{L}_{\rm m})\,\frac{\partial\overline{\eta}}{\partial\varphi^A} - d\Big[(\widehat{V} + \widehat{L}_{\rm m})\,\frac{\partial\,\overline{\eta}}{\partial\,d\varphi^A}\Big] = 0\,. \tag{4.5}$$

The first term vanishes, since $\overline{\eta}$ does not depend on φ^A explicitly, see (3.21). Then the Leibniz rule yields

$$d\left[(\,\widehat{V} + \widehat{L}_{\rm m})\,\frac{\partial\,\overline{\eta}}{\partial\,d\varphi^A}\right] \tag{4.6}$$

$$= \frac{\partial\,\overline{\eta}}{\partial\,d\varphi^A}\,d\,(\widehat{V} + \widehat{L}_{\rm m}) + (\widehat{V} + \widehat{L}_{\rm m})\,\underbrace{d\frac{\partial\,\overline{\eta}}{\partial\,d\varphi^A}}_{\stackrel{(3.20)}{=}0} = 0\,. \tag{4.7}$$

Provided $\varphi^A \neq 0$ and $d\varphi^A \neq 0$, we can conclude that

$$d\,(\widehat{V} + \widehat{L}_{\rm m}) = 0\,, \qquad \text{i.e.,} \qquad \widehat{V} + \widehat{L}_{\rm m} = \text{const}\,. \tag{4.8}$$

The gravitational field equations following from $\delta g_{\alpha\beta}$ and $\delta\Gamma_\alpha{}^\beta$ are *not* disturbed by the existence of φ^A. Hence the usual metric-affine formalism applies in its conventional form (see [2], for recent developments cf. [17–20]),

but the field equation (4.8) for the scalar field quartet φ^A has to be appended. Perhaps surprisingly, it is only one equation since, in addition to (3.23) and (3.24), we trivially have

$$\frac{\delta\overline{\eta}}{\delta\Gamma_\alpha{}^\beta} = 0\,, \quad \frac{\delta\overline{\eta}_A}{\delta\Gamma_\alpha{}^\beta} = 0\,, \quad \text{etc.} \tag{4.9}$$

V. CONCLUSION

We can reproduce the essential features of the Guendelman-Kaganovich theory without the necessity to specify the gravitational first-order Lagrangian other than by the property (4.4).

ACKNOWLEDGMENTS

We would like to thank Eduardo Guendelman and Alex Kaganovich for useful comments and remarks. This research was supported by CONACyT, grants No. 3544–E9311, No. 3898P–E9608, and by the joint German–Mexican project DLR(BMBF)-CONACyT MXI 6.B0A.6A.

APPENDIX: COVARIANT EXTERIOR DERIVATIVE OF THE ϵ-BASIS

For computations with volume elements, it is convenient to introduce the differentials of the $\boldsymbol{\epsilon}$- and the $\widetilde{\epsilon}$-basis. As soon as a linear connection 1-form $\Gamma_\alpha{}^\beta = \Gamma_{i\alpha}{}^\beta dx^i$ is given – there is no need of a metric for that – we find by covariant exterior differentiation of (2.5),

$$D\boldsymbol{\epsilon}_\alpha = T^\mu \wedge \boldsymbol{\epsilon}_{\alpha\mu}\,, \tag{A1a}$$
$$D\boldsymbol{\epsilon}_{\alpha\beta} = T^\mu \wedge \boldsymbol{\epsilon}_{\alpha\beta\mu}\,, \tag{A1b}$$
$$D\boldsymbol{\epsilon}_{\alpha\beta\gamma} = T^\mu \wedge \boldsymbol{\epsilon}_{\alpha\beta\gamma\mu}\,, \tag{A1c}$$
$$D\boldsymbol{\epsilon}_{\alpha\beta\gamma\delta} = 0\,. \tag{A1d}$$

Here $T^\alpha := D\vartheta^\alpha$ is the torsion 2-form. For the $\boldsymbol{\sigma}$-modified $\widetilde{\epsilon}$-basis, the computations run on the same track,

$$D\widetilde{\epsilon}_\alpha = \frac{D\boldsymbol{\sigma}}{\boldsymbol{\sigma}} \wedge \widetilde{\epsilon}_\alpha + T^\mu \wedge \widetilde{\epsilon}_{\alpha\mu}\,, \tag{A2a}$$
$$D\widetilde{\epsilon}_{\alpha\beta} = \frac{D\boldsymbol{\sigma}}{\boldsymbol{\sigma}} \wedge \widetilde{\epsilon}_{\alpha\beta} + T^\mu \wedge \widetilde{\epsilon}_{\alpha\beta\mu}\,, \tag{A2b}$$
$$D\widetilde{\epsilon}_{\alpha\beta\gamma} = \frac{D\boldsymbol{\sigma}}{\boldsymbol{\sigma}} \wedge \widetilde{\epsilon}_{\alpha\beta\gamma} + T^\mu \wedge \widetilde{\epsilon}_{\alpha\beta\gamma\mu}\,, \tag{A2c}$$
$$D\widetilde{\epsilon}_{\alpha\beta\gamma\delta} = \frac{D\boldsymbol{\sigma}}{\boldsymbol{\sigma}} \wedge \widetilde{\epsilon}_{\alpha\beta\gamma\delta}\,. \tag{A2d}$$

the advantage being that this basis is composed of forms of weight 0, i.e., not of densities.

If a metric g is given additionally, then we can take ${}_{\mathbf{0}}\boldsymbol{\sigma}$ as scalar density and find generally,

$$\frac{D\,{}_{\mathbf{0}}\boldsymbol{\sigma}}{{}_{\mathbf{0}}\boldsymbol{\sigma}} = -2\,Q\,, \tag{A3}$$

with the Weyl covector $Q := Q_\gamma{}^\gamma/4$ and the nonmetricity 1-form $Q_{\alpha\beta} := -Dg_{\alpha\beta}$. It is then simple to rewrite (A2) in terms of the metric volume element $\eta := {}_{\mathbf{0}}\boldsymbol{\sigma}\,\boldsymbol{\epsilon}$:

$$D\eta_\alpha = -2Q \wedge \eta_\alpha + T^\mu \wedge \eta_{\alpha\mu}\,, \tag{A4a}$$
$$D\eta_{\alpha\beta} = -2Q \wedge \eta_{\alpha\beta} + T^\mu \wedge \eta_{\alpha\beta\mu}\,, \tag{A4b}$$
$$D\eta_{\alpha\beta\gamma} = -2Q \wedge \eta_{\alpha\beta\gamma} + T^\mu \wedge \eta_{\alpha\beta\gamma\mu}\,, \tag{A4c}$$
$$D\eta_{\alpha\beta\gamma\delta} = -2Q \wedge \eta_{\alpha\beta\gamma\delta}\,. \tag{A4d}$$

These equations turn out to be very helpful in conventional applications. However, in the quartet theory, we have to forget (A4) and to take recourse to ${}_{\mathbf{4}}\boldsymbol{\sigma}$.

Thus, analogously to (A2), we have

$$D\overline{\eta}_\alpha = \frac{D\,{}_{\mathbf{4}}\boldsymbol{\sigma}}{{}_{\mathbf{4}}\boldsymbol{\sigma}} \wedge \overline{\eta}_\alpha + T^\mu \wedge \overline{\eta}_{\alpha\mu}\,, \tag{A5a}$$
$$D\overline{\eta}_{\alpha\beta} = \frac{D\,{}_{\mathbf{4}}\boldsymbol{\sigma}}{{}_{\mathbf{4}}\boldsymbol{\sigma}} \wedge \overline{\eta}_{\alpha\beta} + T^\mu \wedge \overline{\eta}_{\alpha\beta\mu}\,, \tag{A5b}$$
$$D\overline{\eta}_{\alpha\beta\gamma} = \frac{D\,{}_{\mathbf{4}}\boldsymbol{\sigma}}{{}_{\mathbf{4}}\boldsymbol{\sigma}} \wedge \overline{\eta}_{\alpha\beta\gamma} + T^\mu \wedge \overline{\eta}_{\alpha\beta\gamma\mu}\,, \tag{A5c}$$
$$D\overline{\eta}_{\alpha\beta\gamma\delta} = \frac{D\,{}_{\mathbf{4}}\boldsymbol{\sigma}}{{}_{\mathbf{4}}\boldsymbol{\sigma}} \wedge \overline{\eta}_{\alpha\beta\gamma\delta}\,. \tag{A5d}$$

We find

$$\frac{D\,{}_{\mathbf{4}}\boldsymbol{\sigma}}{{}_{\mathbf{4}}\boldsymbol{\sigma}} = \frac{d\,{}_{\mathbf{4}}\boldsymbol{\sigma}}{{}_{\mathbf{4}}\boldsymbol{\sigma}} - \Gamma_\alpha{}^\alpha\,, \tag{A6}$$

but we were not able to find a more compact expression for (A6). However, provided a metric is present besides the scalar quartet and the connection, we can rewrite (A6) as follows:

$$\frac{D\,{}_{\mathbf{4}}\boldsymbol{\sigma}}{{}_{\mathbf{4}}\boldsymbol{\sigma}} = \frac{d\,{}_{\mathbf{4}}\boldsymbol{\sigma}}{{}_{\mathbf{4}}\boldsymbol{\sigma}} - 2Q + d\ln\sqrt{|\det g_{\alpha\beta}|}\,. \tag{A7}$$

[1] See, e.g.: T. Dereli and R. W. Tucker, Class. Quantum Grav. **12**, L31 (1995), gr-qc/9502018.
[2] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne'eman, Physics Reports **258**, 1 (1995).
[3] F. Gronwald, Int. J. of Mod. Phys. **D6**, 263 (1997).
[4] E. I. Guendelman and A. B. Kaganovich, Phys. Rev. **D53**, 7020 (1996), gr-qc/9605026.
[5] E. I. Guendelman and A. B. Kaganovich, Phys. Rev. **D55**, 5970 (1997), gr-qc/9611046.
[6] E. I. Guendelman and A. B. Kaganovich, Phys. Rev. **D56**, 3548 (1997), gr-qc/9702058.

[7] W. L. Burke, *Applied Differential Geometry* (Cambridge University Press, Cambridge, 1985).
[8] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology* (Springer, Berlin, 1982).
[9] B. Laurent, *Introduction to Spacetime, a first course on relativity* (World Scientific, Singapore, 1994).
[10] A. J. Kox, M. J. Klein, and R. Schulmann, *The Collected Papers of Albert Einstein* (Princeton University Press, Princeton, 1996), Vol. 6.
[11] W. Buchmüller and N. Dragon, Phys. Lett. **B207**, 292 (1988).
[12] W. Buchmüller and N. Dragon, Phys. Lett **B223**, 313 (1989).
[13] C. Wetterich, Nucl. Phys. **B302**, 668 (1988).
[14] A. S. Eddington, *The Mathematical Theory of Relativity*, 2nd ed. (Cambridge University Press, Cambridge, 1924).
[15] E. Schrödinger, *Space-Time Structure* (Cambridge University Press, Cambridge, 1960), reprinted with corrections.
[16] D. Catto, M. Francaviglia, and J. Kijowski, Bull. Acad. Polon. Sci. (Phys. Astron.) **28**, 179 (1980).
[17] Y. Ne'eman and F. W. Hehl, Class. Quantum Grav. **14**, A251 (1997), gr-qc/9604047.
[18] Y. N. Obukhov, E. J. Vlachynsky, W. Esser, and F. W. Hehl, Phys. Rev. **D56**, (1997, to be published).
[19] A. Macías, E. W. Mielke, and J. Socorro, Class. Quantum Grav. **15**, (1998, to be published).
[20] A. García *et al.*, Phys. Rev. **D57**, (1998, to be published), gr-qc/9711041.